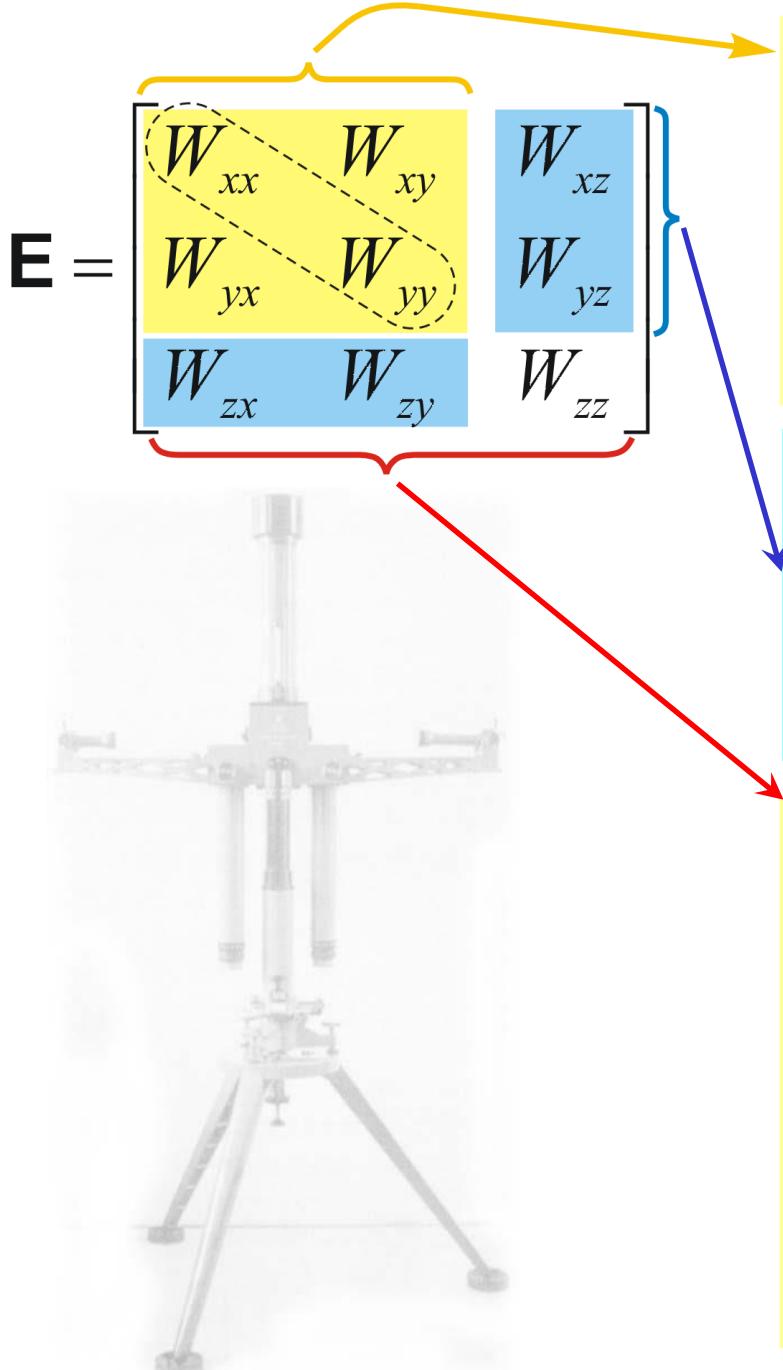


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***Gravity potential as a result  
of series expansion-based  
inversion of torsion balance  
data***

Eötvös 100 Scientific Meeting in Egbell (Gbely, Slovakia)  
2019. October 17.

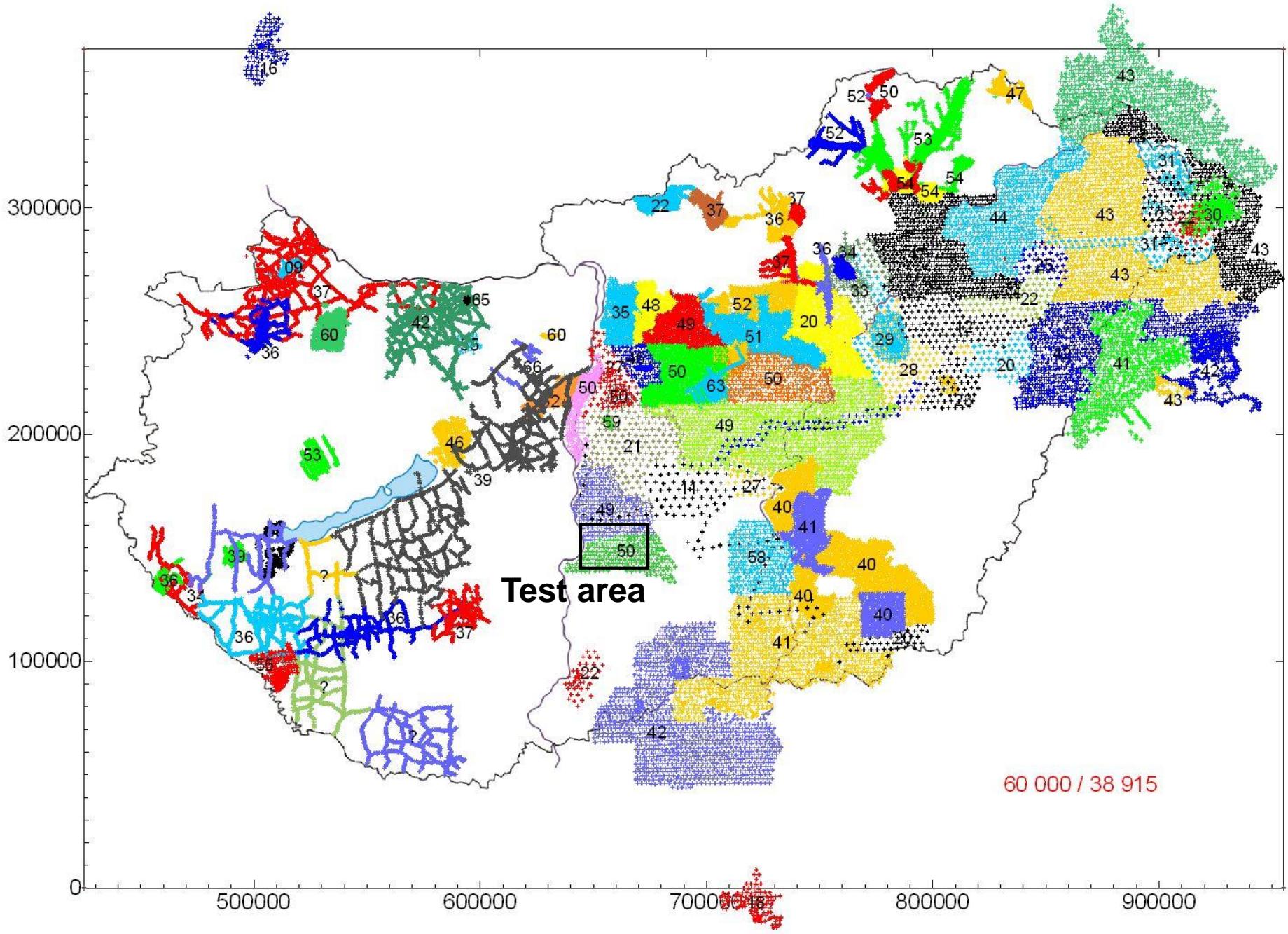


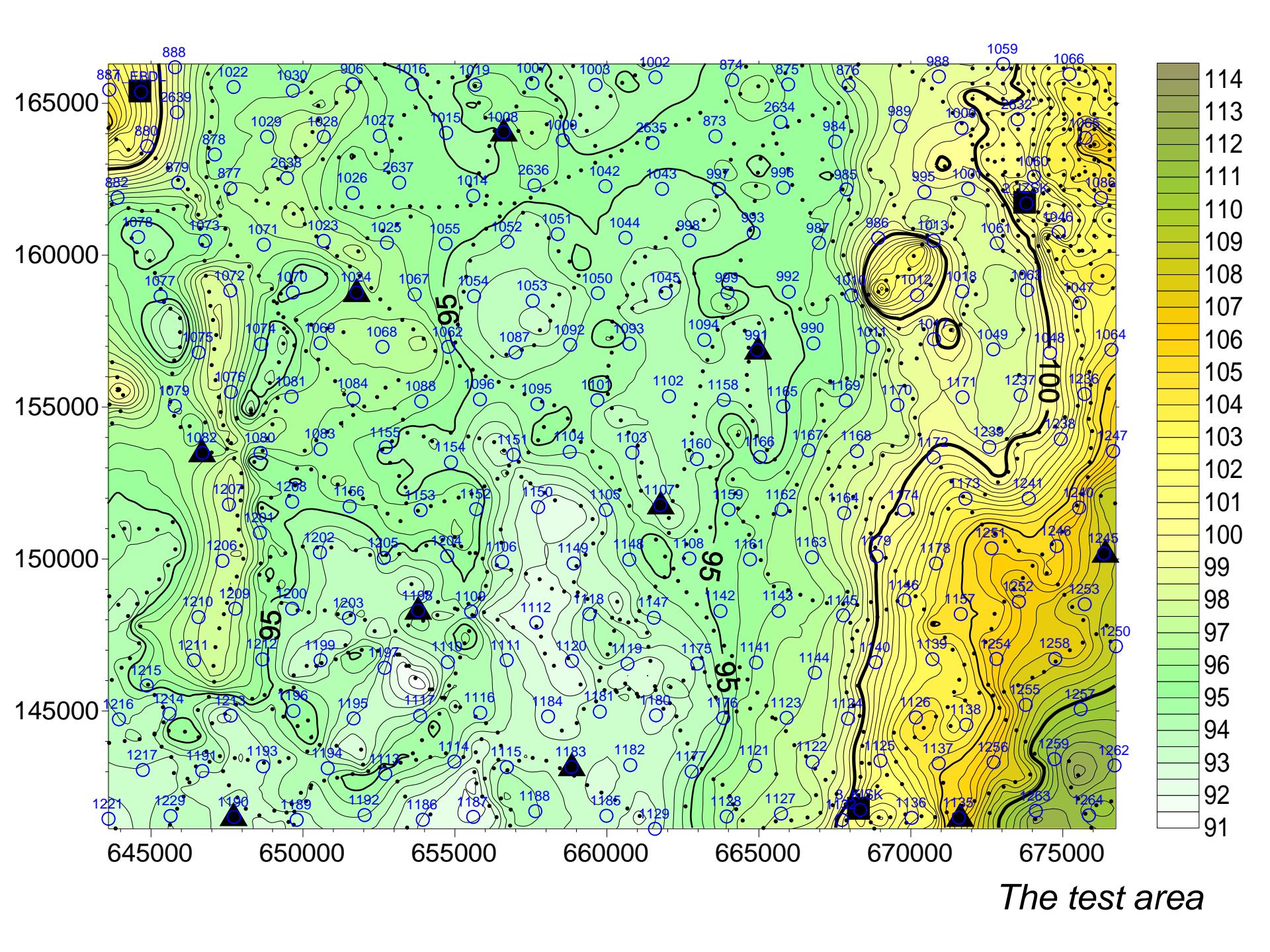
- *interpolation of deflection of the vertical*
- *determination of local geoid forms*

- *determination of gravity and gravity anomalies mainly for geophysical purposes*

- *determination of vertical gradients*

*inversion based reconstruction  
of the 3D potential function*





## 3D inversion algorithm

The gravity potential  $W(x,y,z)$  as an expansion in a series of a known set of basis function  $\Psi_1 \dots \Psi_P$  is

$$W(x, y, z) = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} B_l \Psi_i(x) \Psi_j(y) \Psi_k(z)$$

(where  $l = i + (j-1)*N_y + (k-1)*N_x * N_y$  and  $B_l$  are the unknown expansion coefficients)

### The second derivatives from torsion balance measurements:

$$W_{xy} = \frac{\partial^2 W}{\partial x \partial y} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} B_l \underbrace{\Psi_i'(x) \Psi_j'(y) \Psi_k(z)}_{S_{ql}}$$

$$W_\Delta = W_{yy} - W_{xx} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} B_l \underbrace{\left\{ \Psi_j''(y) \Psi_i(x) - \Psi_j(y) \Psi_i''(x) \right\} \Psi_k(z)}_{Q_{ql}}$$

$$W_{xz} = \frac{\partial^2 W}{\partial x \partial z} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} B_l \underbrace{\Psi_i'(x) \Psi_j(y) \Psi_k'(z)}_{D_{ql}}$$

$$W_{yz} = \frac{\partial^2 W}{\partial y \partial z} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} B_l \underbrace{\Psi_i(x) \Psi_j'(y) \Psi_k'(z)}_{F_{ql}}$$

Introducing the notations:

$$S_{ql} = \Psi_i'(x_q) \Psi_j'(y_q) \Psi_k(z_q)$$

$$Q_{ql} = \left\{ \Psi_j''(y_q) \Psi_i(x_q) - \Psi_j(y_q) \Psi_i''(x_q) \right\} \Psi_k(z_q)$$

$$D_{ql} = \Psi_i'(x_q) \Psi_j(y_q) \Psi_k'(z_q)$$

$$F_{ql} = \Psi_j'(y_q) \Psi_i(x_q) \Psi_k'(z_q)$$

The computed torsion balance data at an arbitrary point ( $q$ ) are:

$$\left. \begin{aligned} W_{xy}^{(q)} &= \sum_{l=1}^M B_l S_{ql} \\ W_{\Delta}^{(q)} &= \sum_{l=1}^M B_l Q_{ql} \\ W_{xz}^{(q)} &= \sum_{l=1}^M B_l D_{ql} \\ W_{yz}^{(q)} &= \sum_{l=1}^M B_l F_{ql} \end{aligned} \right\}$$

where  $M = N_x N_y N_z - 1$  is the number of expansion coefficients,  $S_{ql}, Q_{ql}, D_{ql}, F_{ql}$  are known.

## The first derivatives for the inversion procedure:

$$\left. \begin{aligned} W_z &= \frac{\partial W}{\partial z} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} B_l \Psi_i(x) \Psi_j(y) \Psi_k(z) \\ W_x &= \frac{\partial W}{\partial x} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} B_l \Psi_i(x) \Psi_j(y) \Psi_k(z) \\ W_y &= \frac{\partial W}{\partial y} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} B_l \Psi_i(x) \Psi_j(y) \Psi_k(z) \end{aligned} \right\}$$

Let's introduce the following notations:

$$A_{ql} = \Psi_i(x_q) \Psi_j(y_q) \Psi_k(z_q)$$

$$C_{ql} = \Psi_i(x_q) \Psi_j(y_q) \Psi_k(z_q)$$

$$H_{ql} = \Psi_i(x_q) \Psi_j(y_q) \Psi_k(z_q)$$

Computed data at an arbitrary point ( $q$ ) are :

$$\left. \begin{aligned} W_z^{(q)} &= \sum_{l=1}^M B_l A_{ql} \\ W_x^{(q)} &= \sum_{l=1}^M B_l C_{ql} \\ W_y^{(q)} &= \sum_{l=1}^M B_l H_{ql} \end{aligned} \right\}$$

where  $M = N_x N_y N_z - 1$  is the number of expansion coefficients,  $A_{ql}, C_{ql}, H_{ql}$  are known.

Elements of the deviation vector:

$$\left. \begin{array}{l} e_q^{(1)} = W_{xy_{measured}}^{(q)} - \sum_{l=1}^M B_l S_{ql} \\ e_q^{(2)} = W_{\Delta_{measured}}^{(q)} - \sum_{l=1}^M B_l Q_{ql} \\ e_q^{(3)} = W_{xz_{measured}}^{(q)} - \sum_{l=1}^M B_l D_{ql} \\ e_q^{(4)} = W_{yz_{measured}}^{(q)} - \sum_{l=1}^M B_l F_{ql} \\ e_q^{(5)} = W_{z_{measured}}^{(q)} - \sum_{l=1}^M B_l A_{ql} \\ e_q^{(6)} = W_{x_{measured}}^{(q)} - \sum_{l=1}^M B_l C_{ql} \\ e_q^{(7)} = W_{y_{measured}}^{(q)} - \sum_{l=1}^M B_l H_{ql} \end{array} \right\}$$

The function to be minimized:

$$E = \sum_{s=1}^7 \sum_{k=1}^{N_s} (e_k^{(s)})^2$$

$N_s$  is the number of the  $s$ -th type of data

*Vector of measured data:*

$$\vec{d}^{(measured)} = \{W_{xy}^{(1)}, \dots, W_{xy}^{(N_1)}, W_{\Delta}^{(1)}, \dots, W_{\Delta}^{N_2}, W_{xz}^{(1)}, \dots, W_{xz}^{(N_3)}, \dots, \dots, W_y^{(1)}, \dots, W_y^{(N_7)}, \}$$

*Structure of the Jacobi matrix:*

$$G_{qj} = \begin{cases} S_{qj} & q \leq N_1 \\ \dots & \\ \dots & \\ R_{qj} & \sum_{s=1}^6 N_s < q \leq \sum_{s=1}^7 N_s \end{cases}$$

*Deviation of measured and computed data:*

$$\vec{e} = \vec{d}^{(measured)} - \underline{\underline{G}} \vec{B}$$

and let  $E$  be:

$$E = (\vec{e}, \vec{e}) = \sum_{k=1}^N e_k^2 \quad \text{where: } N = \sum_{s=1}^7 N_s$$

*Incorporating the Laplace equation into the inversion:*

$$\Delta W = W_{xx} + W_{yy} + W_{zz} = 0$$

$$comp \Delta W = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} =$$

$$= \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} B_l \underbrace{\left\{ \Psi_i''(x) \Psi_j(y) \Psi_k(z) + \Psi_i(x) \Psi_j''(y) \Psi_k(z) + \Psi_i(x) \Psi_j(y) \Psi_k''(z) \right\}}_{Lq_l}$$

For the  $q$ -th measurement point:

$$L_{ql} = \Psi_i''(x_q) \Psi_j(y_q) \Psi_k(z_q) + \Psi_i(x_q) \Psi_j''(y_q) \Psi_k(z_q) + \Psi_i(x_q) \Psi_j(y_q) \Psi_k''(z_q)$$

$$comp \Delta W^{(q)} = \sum_{l=1}^M B_l L_{ql}$$

Vector of measured and computed data:

$$meas \mathbf{d} = \left\{ W_{zx}^{(1)}, \dots, W_{zx}^{(N_1)}, W_{zy}^{(1)}, \dots, W_{zy}^{(N_2)}, \dots, W_z^{(1)}, \dots, W_z^{(N_7)}, 0^{(1)}, \dots, 0^{(N_8)} \right\}$$

$$comp \mathbf{d} = \left\{ comp W_{zx}^{(1)}, \dots, comp W_{zx}^{(N_1)}, \dots, comp W_z^{(1)}, \dots, comp W_z^{(N_7)}, comp \Delta W^{(1)}, \dots, comp \Delta W^{(N_8)} \right\}$$

The extended Jacobi-matrix:

$$G_{qj} = \begin{cases} S_{qj} & q \leq N_1 \\ \vdots \\ R_{qj} & \sum_{s=1}^6 N_s < q \leq \sum_{s=1}^7 N_s \\ L_{qj} & \sum_{s=1}^7 N_s < q \leq \sum_{s=1}^8 N_s \end{cases}$$

Based on  $\frac{\partial E}{\partial B_l} = 0 \quad (l = 1, \dots, M)$

The inverse problem  $\underline{\underline{G}}^T \underline{\underline{G}} \vec{B} = \underline{\underline{G}}^T \vec{d}^{(m)}$

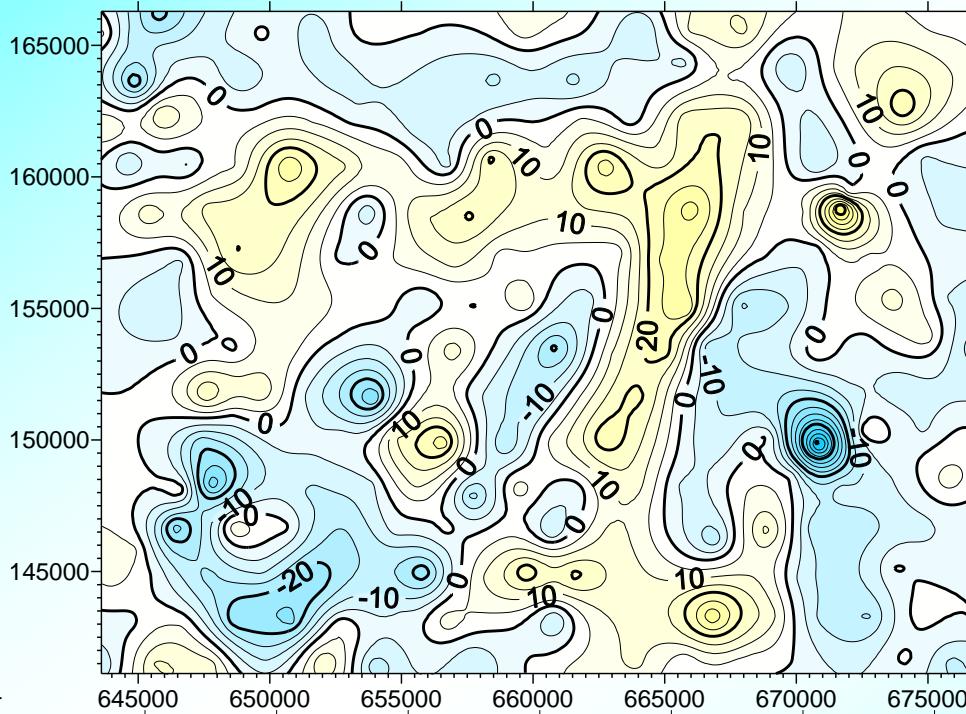
That leads to

$$\vec{B} = (\underline{\underline{G}}^T \underline{\underline{G}})^{-1} \underline{\underline{G}}^T \vec{d}$$

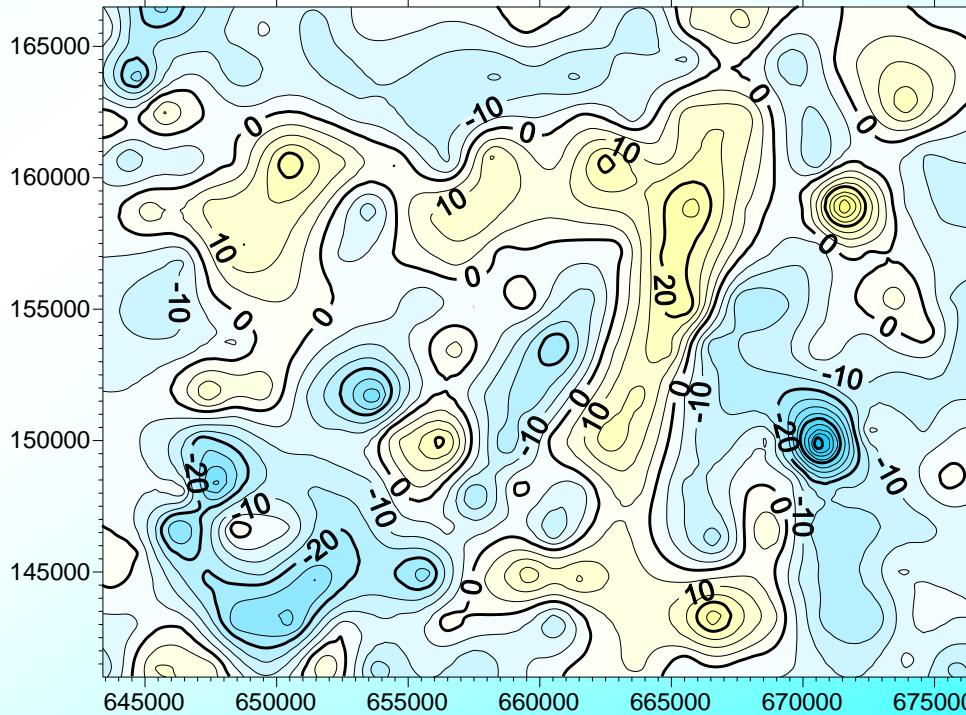
**The inverse problem is linear, by knowing  $\vec{B}$  the full Eötvös tensor can be computed for the investigated area (not just for the measurement points)**

$$\mathbf{E} = \begin{bmatrix} W_{xx} & W_{xy} \\ W_{yx} & W_{yy} \\ W_{zx} & W_{zy} \end{bmatrix} \quad \begin{bmatrix} W_{xz} \\ W_{yz} \\ W_{zz} \end{bmatrix}$$

*measured*  $W_\Delta$

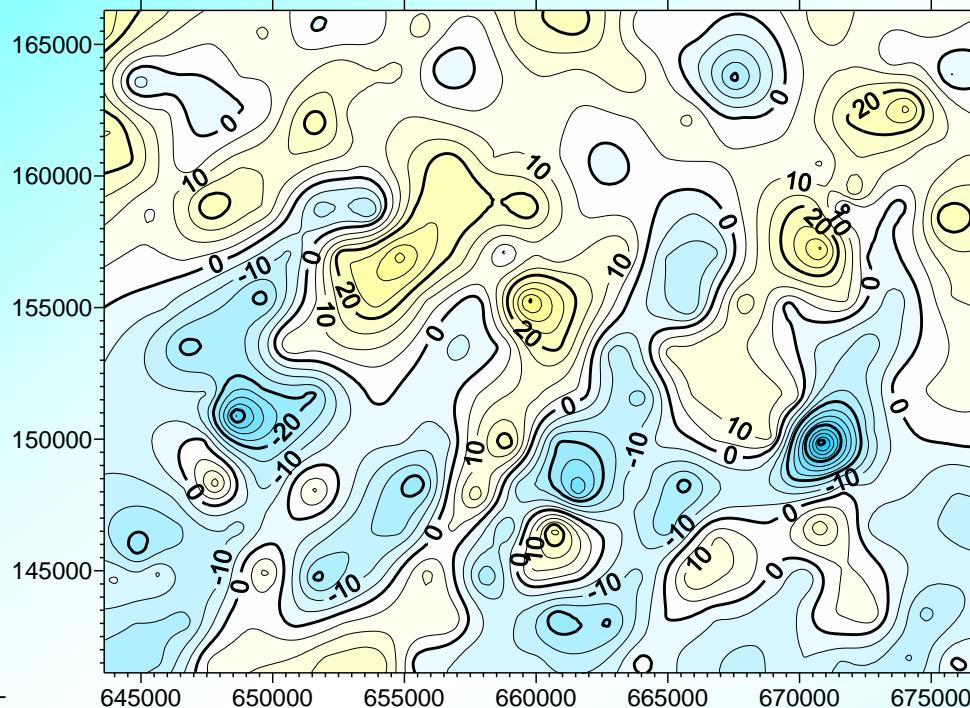


*calculated*  $W_\Delta$

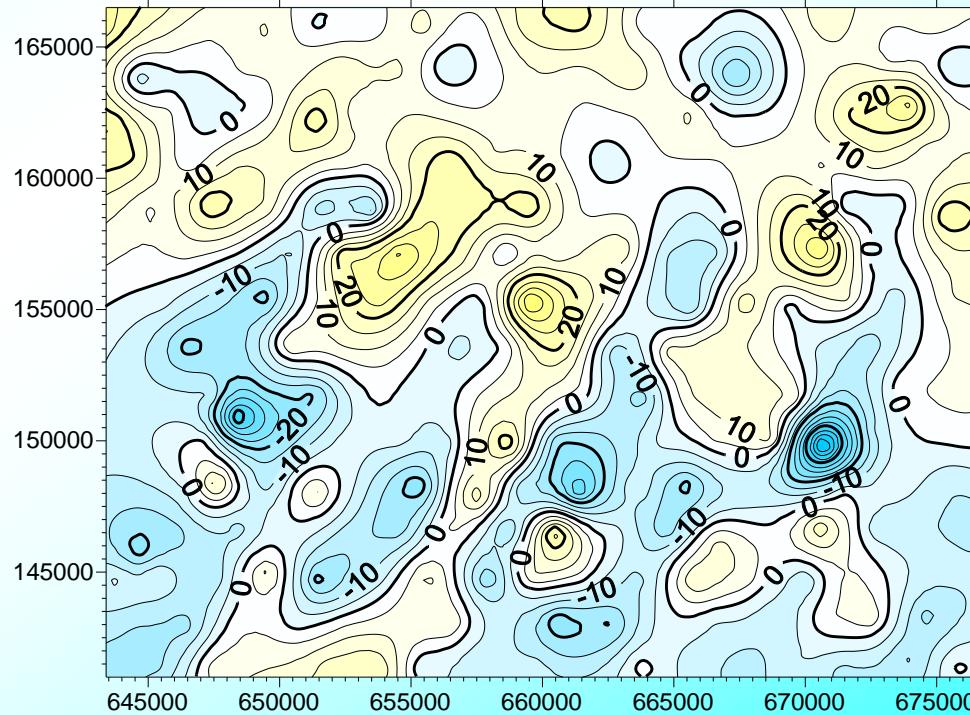


contour interval : 5 E.  
(1E =  $10^{-9}$  s $^2$ )

*measured*  $W_{xy}$

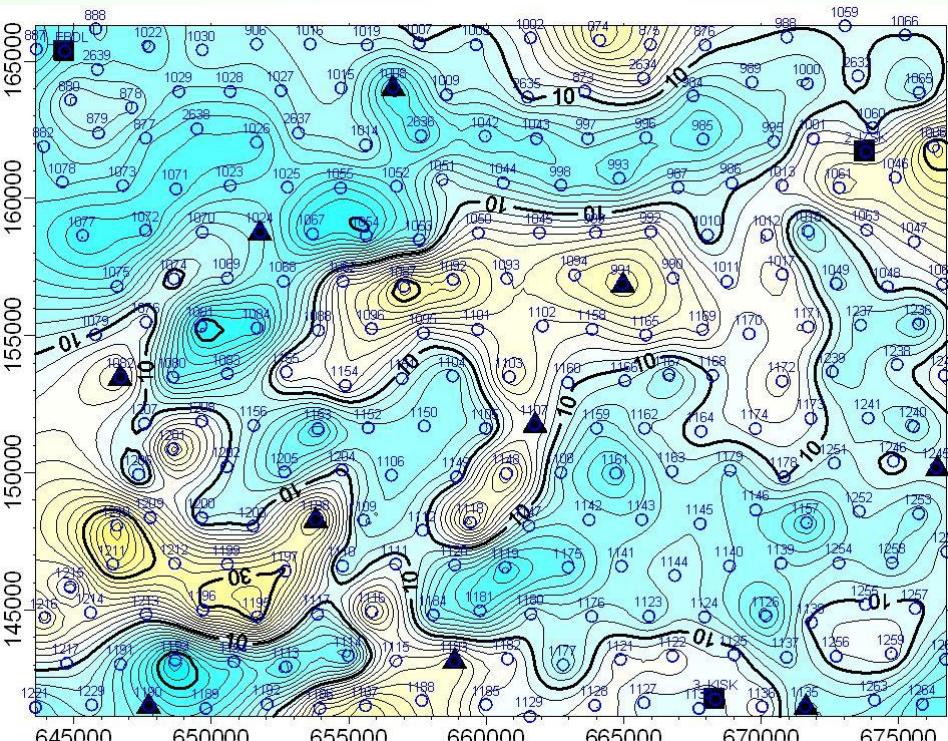


*calculated*  $W_{xy}$

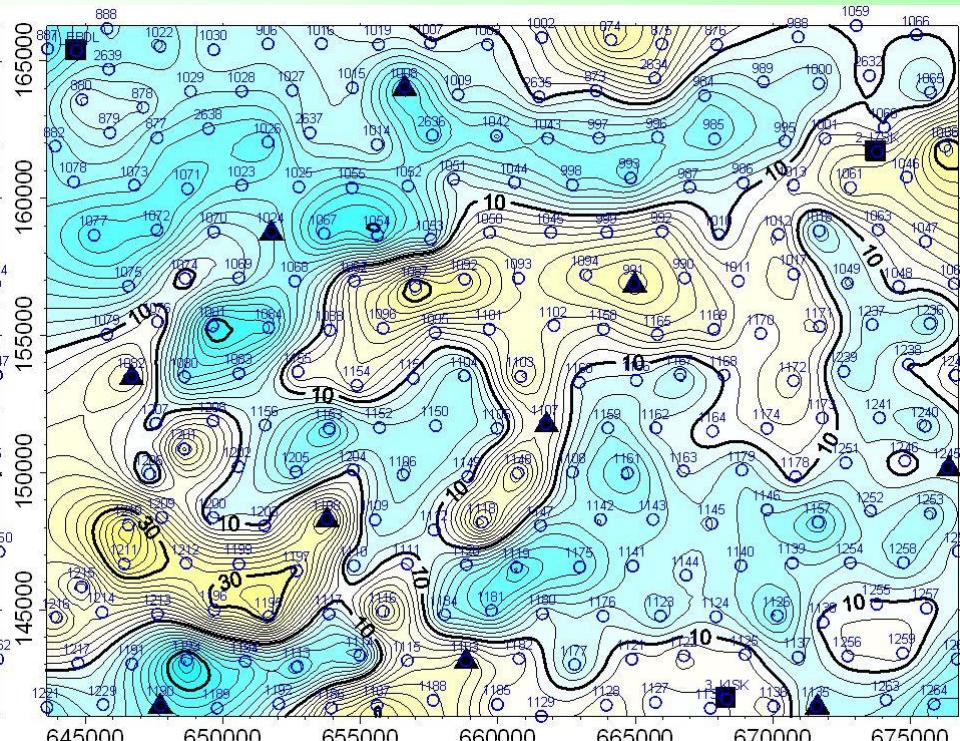


contour interval : 5 E.  
(1E =  $10^{-9} \text{ s}^2$ )

*measured*  
 $W_{zx}$

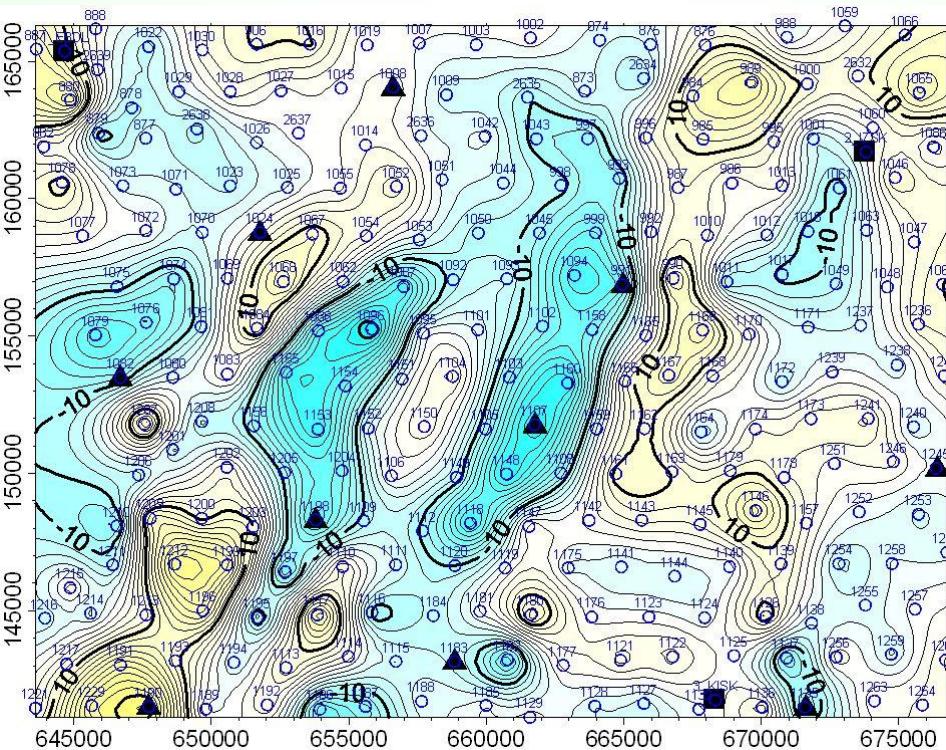


*calculated*  
 $W_{zx}$

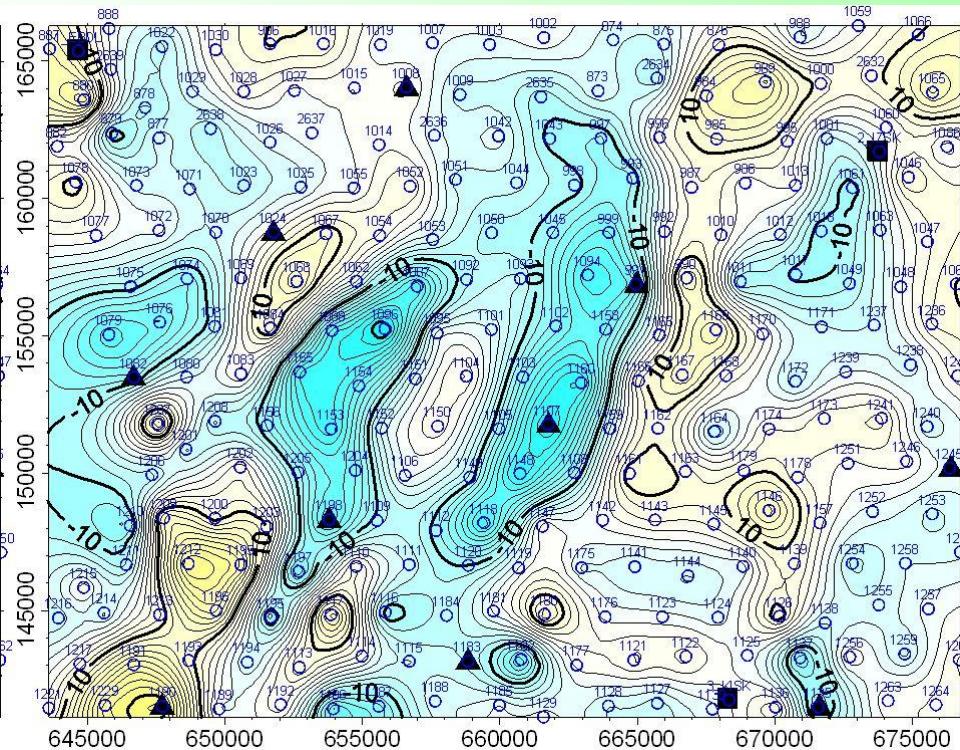


contour interval : 5 E.  
(1E =  $10^{-9} \text{ s}^{-2}$ )

*measured*  $W_{zy}$

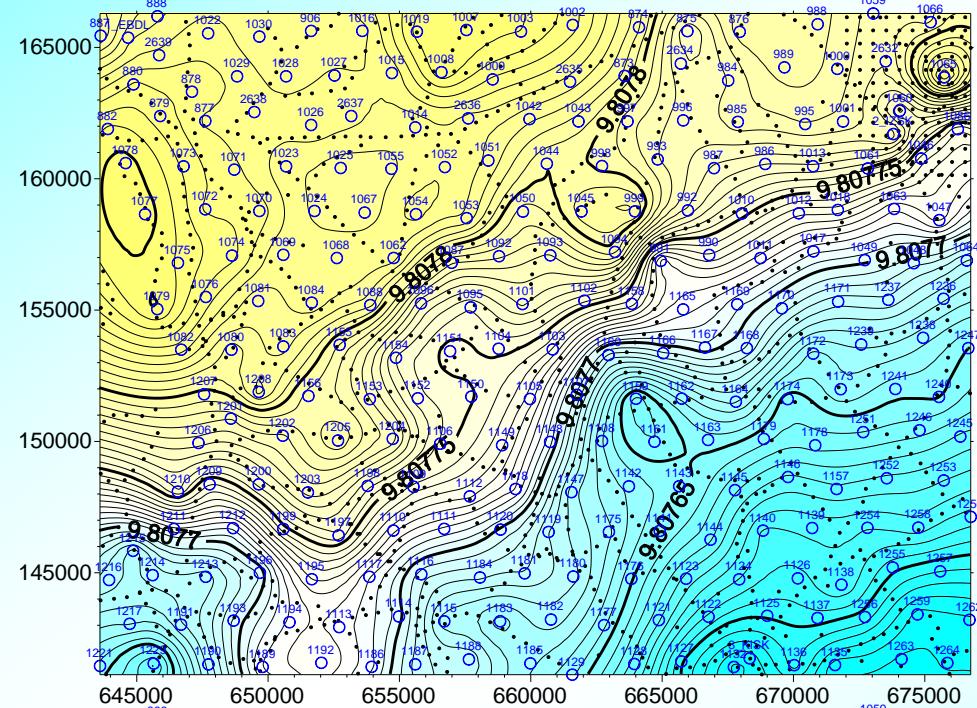


*calculated*  $W_{zy}$

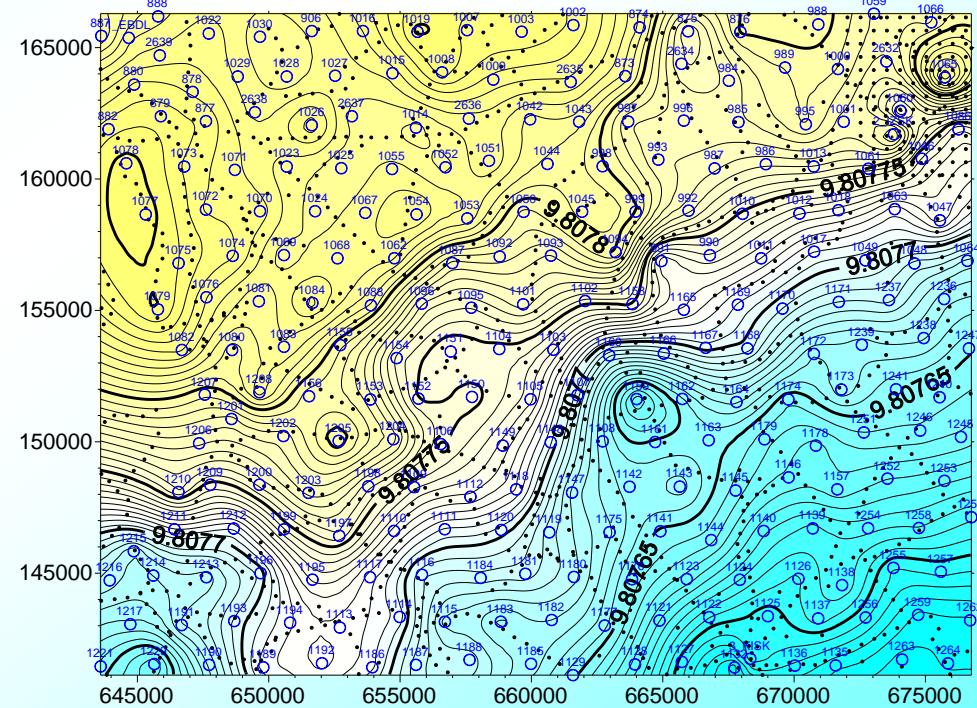


contour interval : 5 E.  
 $(1E = 10^{-9} \text{ s}^{-2})$

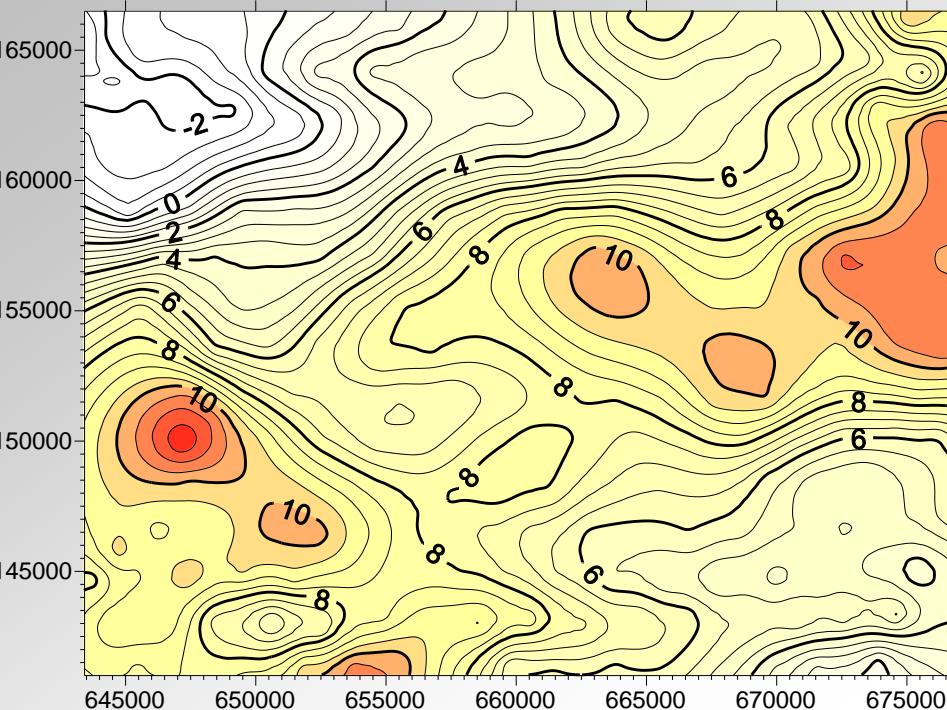
*measured*  $W_z$  (g)



*calculated*  $W_z$  (g)

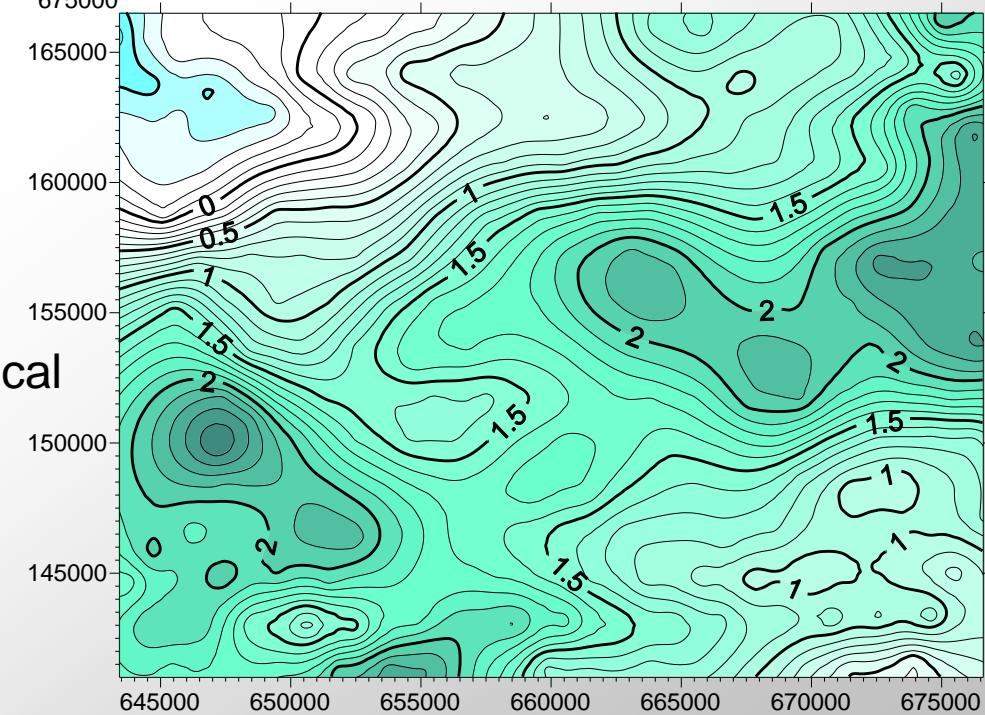


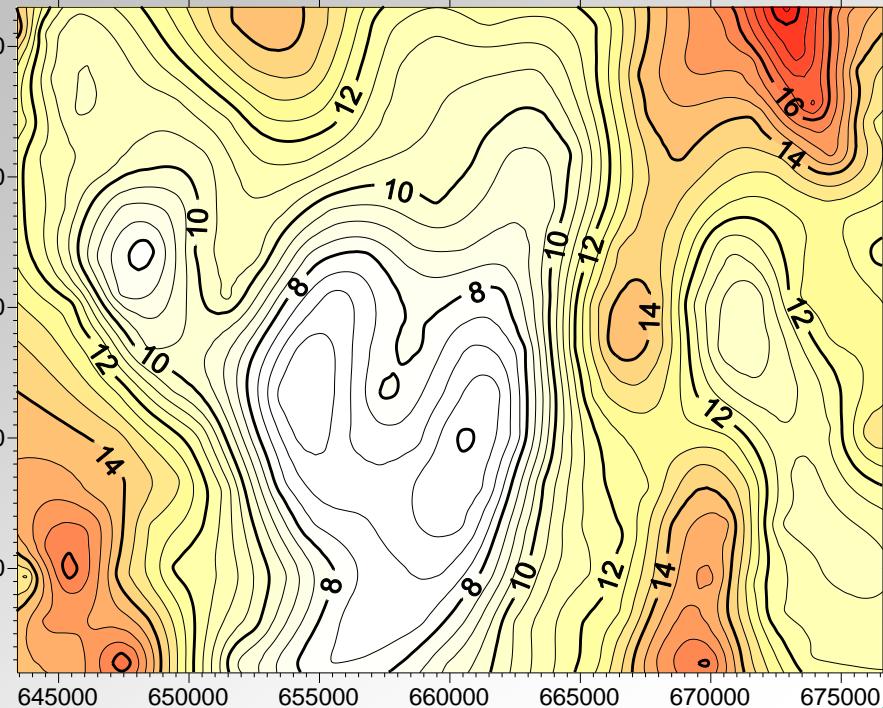
contour interval : 0.01 mGal



$W_x$  ( $g_x$ ) as a result of inversion  
(contour interval 0.5 mGal)

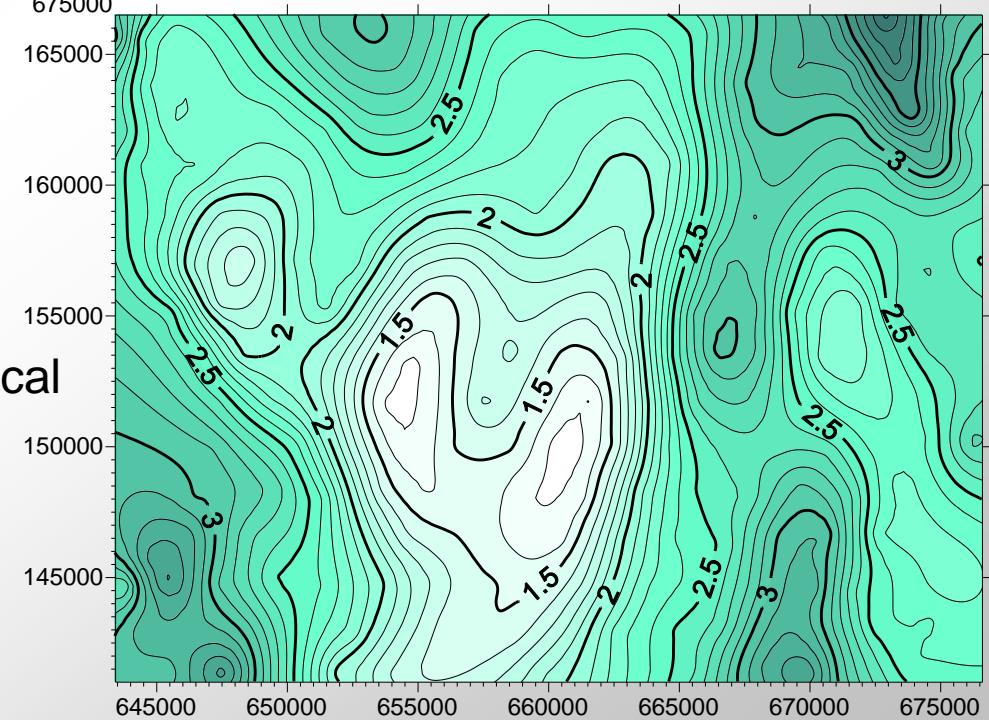
$\xi$  component of the deflection of the vertical  
(contour interval 0.1" )



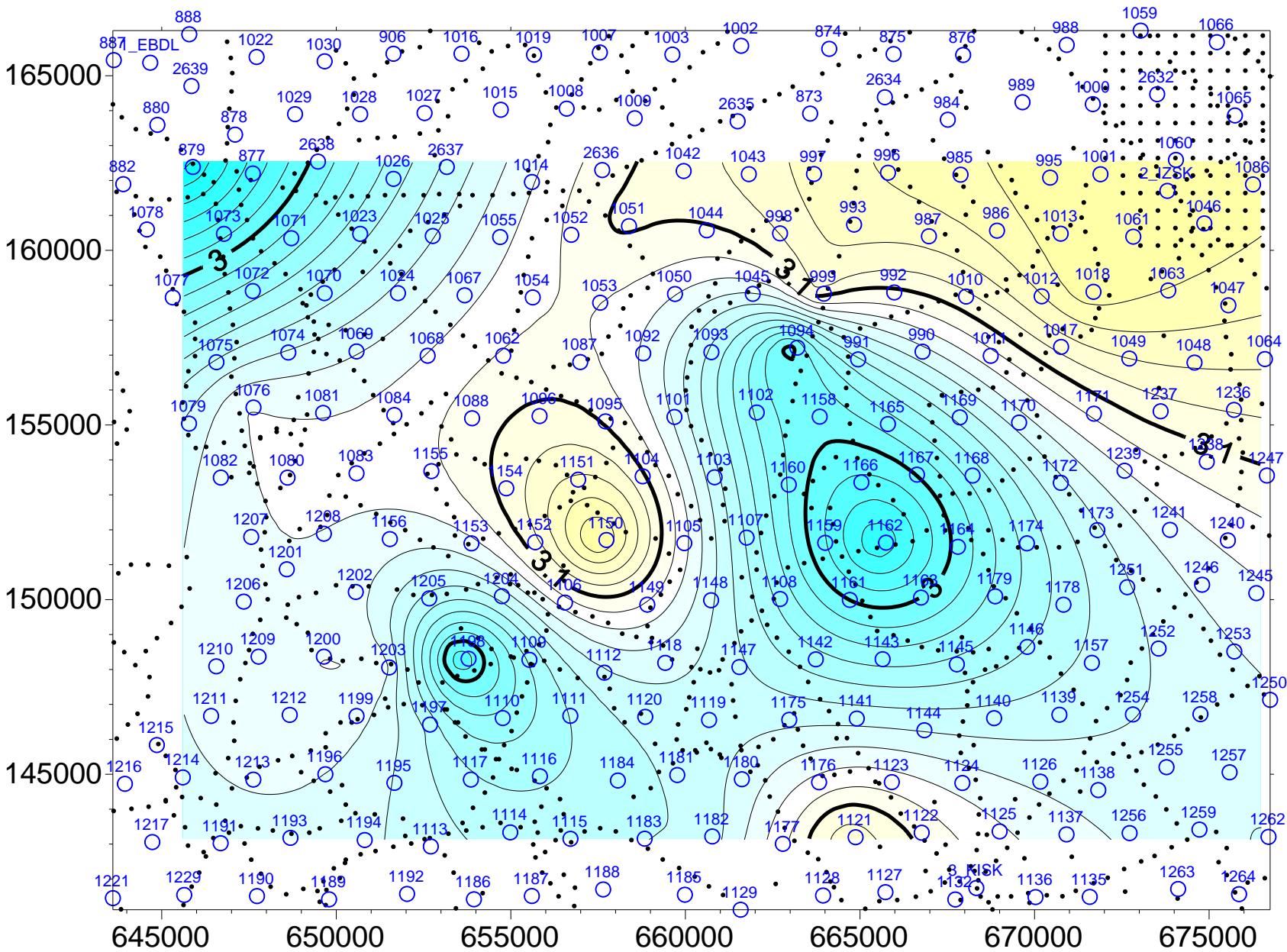


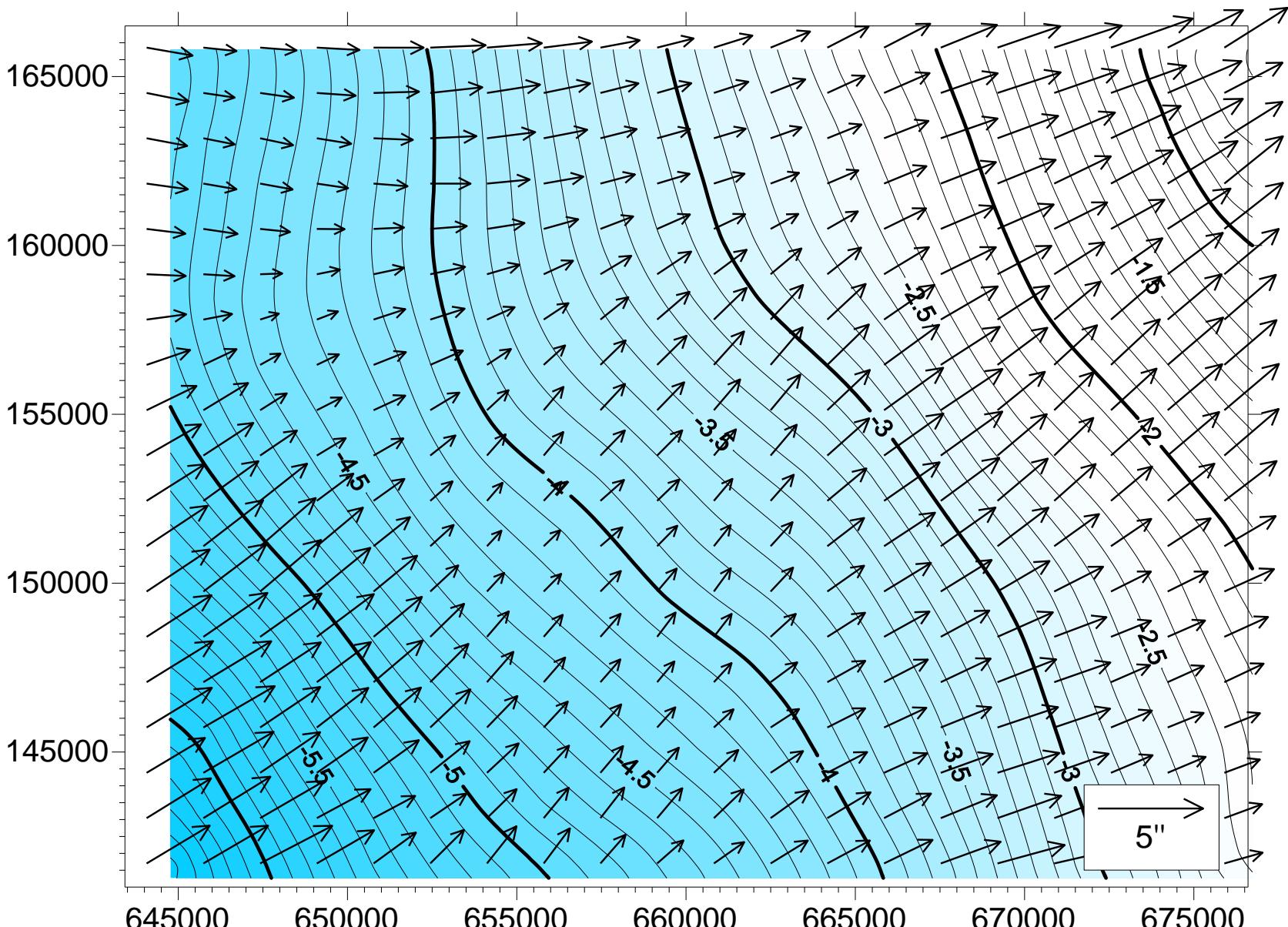
$W_y$  ( $g_y$ ) as a result of inversion  
(contour interval 0.5 mGal)

$\eta$  component of the deflection of the vertical  
(contour interval 0.1")



$W_{zz}$  vertical gradient as a result of inversion, contour interval: 10 E





Thank you for your attention!